NONREFLECTING STATIONARY SETS IN $\mathcal{P}_\kappa\lambda$

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Abstract. Let κ be a regular uncountable cardinal and $\lambda \geq \kappa^+$. The principle of stationary reflection for $\mathcal{P}_\kappa\lambda$ has been successful in settling problems of infinite combinatorics in the case $\kappa = \omega_1$. For a greater κ the principle is known to fail at some λ. This note shows that it fails at every λ if κ is the successor of a regular uncountable cardinal or κ is countably closed.

1. Introduction

In [6] Foreman, Magidor and Shelah introduced the following principle for $\lambda \geq \omega_2$: If S is a stationary subset of $\mathcal{P}_{\omega_1}\lambda$, then $S \cap \mathcal{P}_{\omega_1}A$ is stationary in $\mathcal{P}_{\omega_1}A$ for some $\omega_1 \subset A \subset \lambda$ of size ω_1. Let us call the principle stationary reflection for $\mathcal{P}_{\omega_1}\lambda$. It follows from Martin's Maximum (see [6]) and holds in the Lévy model where ω_2 was supercompact in the ground model (see [2]). See [3, 15, 17, 18] for recent applications of reflection principles for stationary sets in $\mathcal{P}_{\omega_1}\lambda$.

What if ω_1 is replaced by a higher regular cardinal? Feng and Magidor [4] proved that the corresponding statement for $\mathcal{P}_{\omega_2}\lambda$ is false at some large enough λ. Their argument (see also [2]) showed in effect that stationary reflection for $\mathcal{P}_\kappa\lambda$ at some large enough λ implies the presaturation of the club filter on κ for a successor cardinal κ, which is known to be false if in addition $\kappa \geq \omega_2$ by [11].

Foreman and Magidor [5] extended the Feng–Magidor result for every regular cardinal $\kappa \geq \omega_2$, although they proved only the case $\kappa = \omega_2$. We present below what was proved in effect and in §4 its proof of our own:

Theorem 1. *Stationary reflection for $\mathcal{P}_\kappa\lambda$ fails at every $\lambda \geq 2^{\kappa^+}$ if $\kappa \geq \omega_2$ is regular.*

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See [13] for a further example of nonreflection, which is based on pcf theory [12]. This note addresses the problem whether stationary reflection for $\mathcal{P}_\kappa\lambda$ fails *everywhere*, i.e. at every $\lambda \geq \kappa^+$. Specifically we prove

Theorem 2. *Stationary reflection for $\mathcal{P}_\kappa\lambda$ fails everywhere if $\nu < \kappa$ are both regular uncountable and $\mathrm{cf}(\nu,\gamma) < \kappa$ for $\nu < \gamma < \kappa$.*

Here $\mathrm{cf}(\nu,\gamma)$ is the smallest size of unbounded subsets of $\mathcal{P}_\nu\gamma$. The last condition in Theorem 2 holds if $\kappa = \nu^+$ or if $\nu = \omega_1$ and $\gamma^\omega < \kappa$ for $\gamma < \kappa$. In §3 we prove Theorem 2 in much greater generality.

2. Preliminaries

For background material we refer the reader to [7]. Throughout the paper, κ and ν stand for a regular cardinal $\geq \omega_1$ and $\mu < \lambda$ a cardinal $\geq \kappa$. We write S^ν_κ for $\{\gamma < \kappa : \mathrm{cf}\,\gamma = \nu\}$. Let A be a set of ordinals. The set of limit points of A is denoted $\lim A$. It is easy to see $|\lim A| \leq |A|$. A is called σ-closed if $\gamma \in A$ for $\gamma \in \lim A$ of cofinality ω. Let $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$. We write $C(f)$ for $\{x \in \mathcal{P}_\kappa\lambda : \bigcup f``[x]^{<\omega} \subset x\}$. For $x \in \mathcal{P}_\kappa\lambda$ the smallest superset of x in $C(f)$ is denoted $\mathrm{cl}_f\, x$.

Stationary reflection for $\mathcal{P}_\kappa\lambda$ states that if S is a stationary subset of $\mathcal{P}_\kappa\lambda$, then $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size κ. It is easily seen that stationary reflection for $\mathcal{P}_\kappa\lambda$ implies one for $\mathcal{P}_\kappa\mu$. Hence stationary reflection for $\mathcal{P}_\kappa\lambda$ fails everywhere iff it fails at $\lambda = \kappa^+$.

Let S be a stationary subset of $\mathcal{P}_\kappa\lambda$. S is called nonreflecting if it witnesses the failure of stationary reflection, i.e. $S \cap \mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for $\kappa \subset A \subset \lambda$ of size κ. More generally S is called μ-nonreflecting if $S \cap \mathcal{P}_\kappa A$ is nonstationary in $\mathcal{P}_\kappa A$ for $\mu \subset A \subset \lambda$ of size μ.

We write $[\lambda]^\mu$ for $\{x \subset \lambda : |x| = \mu\}$. A filter F on $[\lambda]^\mu$ is called fine if it is μ^+-complete and $\{x \in [\lambda]^\mu : \alpha \in x\} \in F$ for $\alpha < \lambda$. The specific example relevant to us was introduced in [10]:

Lemma 1. *A fine filter on $[\lambda]^\mu$ is generated by the sets of the form $\{\bigcup_{n<\omega} A_n : \{A_n : n < \omega\} \subset [\lambda]^\mu \wedge \forall n < \omega(\varphi(\langle A_k : k < n\rangle) \subset A_n)\}$, where $\varphi : ([\lambda]^\mu)^{<\omega} \to [\lambda]^\mu$.*

We need an analogue [9] of Ulam's theorem in our context:

Lemma 2. *$[\lambda]^\mu$ splits into λ disjoint F-positive sets if F is a fine filter on $[\lambda]^\mu$.*

Proof. It suffices to split X F-positive into ν disjoint F-positive sets for $\mu < \nu \leq \lambda$ regular. Fix a bijection $\pi_x : \mu \to x$ for $x \in X$. Set

$X_{\gamma\xi} = \{x \in X : \pi_x(\xi) = \gamma\}$ for $\gamma < \nu$ and $\xi < \mu$. Then $\bigcup_{\xi<\mu} X_{\gamma\xi} = \{x \in X : \gamma \in x\}$ is F-positive for $\gamma < \nu$. Hence for $\gamma < \nu$ we have $\xi < \mu$ such that $X_{\gamma\xi}$ is F-positive, since F is μ^+-complete. Thus we have F-positive sets $\{X_{\gamma\xi} : \gamma \in A\} \subset \mathcal{P}X$ for some $A \in [\nu]^\nu$ and $\xi < \mu$, which are mutually disjoint, as desired. □

3. Main Theorem

This section is devoted to the main result of this paper. Like the proof [11] of a diamond principle for some $\mathcal{P}_{\omega_1}\lambda$ (see also [16]), our argument originates from nonstructure theory [14].

Throughout the section, let $\nu < \kappa$ be regular cardinals $\geq \omega_1$ and $\mu < \lambda$ cardinals $\geq \kappa$. Recall from [12] $\operatorname{cov}(\lambda, \mu^+, \mu^+, \nu) = \lambda$ iff $\{\bigcup_{\alpha \in a} E_\alpha : a \in \mathcal{P}_\nu\lambda\}$ is unbounded in $[\lambda]^\mu$ for some $\{E_\alpha : \alpha < \lambda\} \subset [\lambda]^\mu$. It is easy to see $\operatorname{cov}(\mu^+, \mu^+, \mu^+, \nu) = \mu^+$.

For the moment assume further $\operatorname{cf}(\nu, \gamma) < \kappa$ for $\nu < \gamma < \kappa$. Inductively we have $\{c_\xi : \xi < \kappa\} \subset \mathcal{P}_\nu\kappa$ and $g : \kappa \to \kappa$ so that $\{c_\xi : \xi < g(\gamma)\}$ is unbounded in $\mathcal{P}_\nu\gamma$ for $\nu \leq \gamma < \kappa$. Then $T = \{\gamma \in S^\nu_\kappa : g``\gamma \subset \gamma\}$ is stationary in κ and $\{c_\xi : \xi < \gamma\}$ is unbounded in $\mathcal{P}_\nu\gamma$ for $\gamma \in T$. Hence Theorem 2 follows from the case $\lambda = \mu^+ = \kappa^+$ of

Theorem 3. *Assume* $\operatorname{cov}(\lambda, \mu^+, \mu^+, \nu) = \lambda$, $\{c_\xi : \xi < \mu\} \subset \mathcal{P}_\nu\mu$, T *is a stationary subset of* $\mathcal{P}_\kappa\mu$ *of size* μ *and* $\{c_\xi : \xi \in z\}$ *is unbounded in* $\mathcal{P}_\nu z$ *for* $z \in T$. *Then* $\mathcal{P}_\kappa\lambda$ *has a* μ*-nonreflecting stationary subset.*

Proof. Let $\{E_\alpha : \alpha < \lambda\} \subset [\lambda]^\mu$ witness $\operatorname{cov}(\lambda, \mu^+, \mu^+, \nu) = \lambda$. Define $e : \lambda \times \mu \to \lambda$ so that $E_\alpha = e``\{\alpha\} \times \mu$. Hence for $A \in \mathcal{P}_{\mu^+}\lambda$ we have $a \in \mathcal{P}_\nu\lambda$ with $A \subset e``a \times \mu$. Let F be the filter on $[\lambda]^\mu$ as defined in Lemma 1. Lemma 2 allows us to split $[\lambda]^\mu$ into μ disjoint F-positive sets $\{X_z : z \in T\}$.

Set $S = \{x \in \mathcal{P}_\kappa\lambda : e``x \times (x \cap \mu) \subset x \wedge x \cap \mu \in T \wedge \exists b \in \mathcal{P}_\nu x(x \subset e``b \times \mu = e``x \times \mu \in X_{x\cap\mu})\}$.

Claim. *S is stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. Fix $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$. We may assume $e``x \times (x \cap \mu) \subset x$ for $x \in C(f)$. For $z \in T$ consider the following game $\mathcal{G}(z)$ of length ω between two players *I* and *II*:

At round n *I* plays $\mu \subset A_n \subset \lambda$ of size μ. Then *II* plays a triple of $b_n \in \mathcal{P}_\nu\lambda$, a bijection $\pi_n : \mu \to e``b_n \times \mu$ and $x_n \in C(f)$ such that $b_n \subset x_n = \pi_n``(x_n \cap \mu)$. We further require $A_n \subset e``b_n \times \mu \subset e``x_n \times \mu \subset A_{n+1}$ and $x_n \subset x_{n+1}$. Finally we let *II* win iff $x_n \cap \mu = z$ for $n < \omega$.

Set $T' = \{z \in T : II$ has no winning strategy in $\mathcal{G}(z)\}$.

Subclaim. *T' is nonstationary in $\mathcal{P}_\kappa\mu$.*

Proof. Suppose otherwise. For $z \in T'$ we have a winning strategy τ_z for I in $\mathcal{G}(z)$, since the game is closed for II, hence determined. By induction on $n < \omega$ build b_n, π_n and $\{x^z_n : z \in T'\}$ so that $\langle(b_n, \pi_n, x^z_n) : n < \omega\rangle$ is a play of II in $\mathcal{G}(z)$ against τ_z as follows:

Since $|T'| \leq |T| = \mu$, we have in ω steps $b_n \in \mathcal{P}_\nu\lambda$ such that $\bigcup_{z\in T'} \tau_z(\langle(b_k, \pi_k, x^z_k) : k < n\rangle) \subset e``b_n \times \mu$, $b_n \subset e``b_n \times \mu$ and $e``b_n \times \mu$ is closed under $f$. Next fix a bijection $\pi_n : \mu \to e``b_n \times \mu$. Note that $x^z_{n-1} = \pi_{n-1}``(x^z_{n-1} \cap \mu) \subset e``b_{n-1} \times \mu \subset \tau_z(\langle(b_k, \pi_k, x^z_k) : k < n\rangle) \subset e``b_n \times \mu$ for $z \in T'$. Hence we have $x^z_{n-1} \cup b_n \subset x^z_n \subset e``b_n \times \mu$ such that $\pi_n``(x^z_n \cap \mu) = x^z_n \in C(f)$, since $b_n \subset e``b_n \times \mu$ and $e``b_n \times \mu$ is closed under $f$. If possible, we further require $x^z_n \cap \mu = z$, in which case we have $x^z_n = \pi_n``z$.

Set $b = \bigcup_{n<\omega} b_n \in \mathcal{P}_\nu\lambda$ and $E = e``b \times \mu \in [\lambda]^\mu$. Then $b \subset E$ by $b_n \subset e``b_n \times \mu$. Since $e``b_n \times \mu \subset e``b_{n+1} \times \mu$ are closed under f, so is E. Also $\mu \subset \bigcup_{z\in T'} \tau_z(\emptyset) \subset e``b_0 \times \mu \subset E$. Since $T'$ is stationary in $\mathcal{P}_\kappa\mu$, we have $b \subset x \subset E$ such that $x \in C(f)$, $\pi_n``(x \cap \mu) = x \cap e``b_n \times \mu$ for $n < \omega$ and $x \cap \mu \in T'$.

Set $z = x \cap \mu$. Since $\mu \subset e``b_0 \times \mu \subset e``b_n \times \mu$, it is easily seen that $x \cap e``b_n \times \mu = \pi_n``z$ meets the requirements for x^z_n. Hence $x^z_n = x \cap e``b_n \times \mu$ and $x^z_n \cap \mu = x \cap \mu = z$ for $n < \omega$. Thus II wins against τ_z with the play $\langle(b_n, \pi_n, x^z_n) : n < \omega\rangle$, which contradicts that τ_z is a winning strategy for I in $\mathcal{G}(z)$, as desired. □

Fix $z \in T - T'$ with a winning strategy τ for II in $\mathcal{G}(z)$. Define $\varphi : ([\lambda]^\mu)^{<\omega} \to [\lambda]^\mu$ by $\varphi(\emptyset) = \mu$ and $\varphi(s) = e``x \times \mu$, where $\tau(s) = (b, \pi, x)$. Since X_z is F-positive, $\bigcup_{n<\omega} A_n \in X_z$ for some $\{A_n : n < \omega\} \subset [\lambda]^\mu$ such that $\varphi(\langle A_k : k < n\rangle) \subset A_n$ for $n < \omega$. Set $(b_n, \pi_n, x_n) = \tau(\langle A_k : k \leq n\rangle)$ for $n < \omega$. Then $\langle A_n : n < \omega\rangle$ is a play of I in $\mathcal{G}(z)$ against τ, since $\mu = \varphi(\emptyset) \subset A_0$ and $e``x_n \times \mu = \varphi(\langle A_k : k \leq n\rangle) \subset A_{n+1}$.

Set $x = \bigcup_{n<\omega} x_n$. Since $\{x_n : n < \omega\} \subset C(f)$ is increasing, we have $x \in C(f)$, hence $e``x \times (x \cap \mu) \subset x$. Also $x \cap \mu = z \in T$ by $x_n \cap \mu = z$. Note that $b_n \in \mathcal{P}_\nu\lambda$, $b_n \subset x_n = \pi_n``(x_n \cap \mu) \subset e``b_n \times \mu$ and $A_n \subset e``b_n \times \mu \subset e``x_n \times \mu \subset A_{n+1}$ for $n < \omega$. Hence $b = \bigcup_{n<\omega} b_n \in \mathcal{P}_\nu x$. Also $x \subset e``b \times \mu = e``x \times \mu = \bigcup_{n<\omega} A_n \in X_z = X_{x\cap\mu}$. Thus we have $x \in S \cap C(f)$, as desired. □

Claim. *S is μ-nonreflecting.*

Proof. Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\mu \subset A \subset \lambda$ of size μ. Then $\{x \in \mathcal{P}_\kappa A : e``x \times (x \cap \mu) \subset x\}$ is unbounded in $\mathcal{P}_\kappa A$, hence $e``A \times \mu \subset A$. Moreover $A = e``a \times \mu$ for some $a \in \mathcal{P}_\nu A$:

Fix a bijection $\pi : \mu \to A$. Then $U = \{x \cap \mu : \pi``(x \cap \mu) = x \in S \cap \mathcal{P}_\kappa A\}$ is a stationary subset of T. For $z \in U$ we have $b \in \mathcal{P}_\nu z$ and

$\xi \in z$ such that $\pi“z \subset e“(\pi“z) \times \mu = e“(\pi“b) \times \mu \subset e“(\pi“c_\xi) \times \mu$, since $\pi“z \in S$ and $\{c_\xi : \xi \in z\}$ is unbounded in $\mathcal{P}_\nu z$. Take $\xi < \mu$ and $U^* \subset U$ stationary in $\mathcal{P}_\kappa\mu$ so that $\pi“z \subset e“(\pi“c_\xi) \times \mu$ for $z \in U^*$. Since $\{\pi“z : z \in U^*\}$ is stationary in $\mathcal{P}_\kappa A$, $A = \bigcup_{z \in U^*} \pi“z \subset e“(\pi“c_\xi) \times \mu \subset e“A \times \mu \subset A$. Hence $A = e“(\pi“c_\xi) \times \mu$ and $\pi“c_\xi \in \mathcal{P}_\nu A$, as desired.

For $i = 0, 1$ take $a \subset x^i \in S \cap \mathcal{P}_\kappa A$ so that $x^i \cap \mu$ disagrees with each other. Then $A = e“a \times \mu \subset e“x^i \times \mu \subset e“A \times \mu \subset A$. Hence $A = e“x^i \times \mu \in X_{x^i \cap \mu}$ by $x^i \in S$, which contradicts that $X_{x^i \cap \mu}$ is disjoint from each other, as desired. $\square$

Therefore S is the desired set. $\square$

Let us derive another

Corollary. *$\mathcal{P}_\kappa\lambda$ has a κ^+-nonreflecting stationary subset if $\lambda \geq \kappa^{++}$ and $\mathrm{cf}(\nu, \gamma) < \kappa$ for $\nu < \gamma < \kappa$.*

Proof. It suffices to prove the case $\lambda = \kappa^{++}$ by checking the conditions of Theorem 3 for $\lambda = \mu^+ = \kappa^{++}$.

For $\gamma < \mu = \kappa^+$ fix a club set $T_\gamma \subset \mathcal{P}_\kappa\gamma$ of size κ and for $z \in \bigcup_{\gamma<\mu} T_\gamma$ an unbounded set $C_z \subset \mathcal{P}_\nu z$ of size $< \kappa$. Set $\{c_\xi : \xi < \mu\} = \bigcup\{C_z : z \in \bigcup_{\gamma<\mu} T_\gamma\}$. Then $T = \{z \in \bigcup_{\gamma<\mu} T_\gamma : \{c_\xi : \xi \in z\}$ is unbounded in $\mathcal{P}_\nu z\}$ has size μ. We claim that T is stationary in $\mathcal{P}_\kappa\mu$.

Fix $f : [\mu]^{<\omega} \to \mathcal{P}_\kappa\mu$. We have $\gamma < \mu$ of cofinality κ such that $\bigcup f“[\gamma]^{<\omega} \cup \bigcup_{\xi<\gamma} c_\xi \subset \gamma$ and $C_y \subset \{c_\xi : \xi < \gamma\}$ for $y \in \bigcup_{\beta<\gamma} T_\beta$. Build an increasing and continuous sequence $\{z_\alpha : \alpha < \nu\} \subset T_\gamma$ so that $\bigcup f“[z_\alpha]^{<\omega} \cup \bigcup\{c_\xi : \xi \in z_\alpha\} \subset z_{\alpha+1}$ and $C_y \subset \{c_\xi : \xi \in z_{\alpha+1}\}$ for some $z_\alpha \subset y \in \bigcup_{\beta<\gamma} T_\beta$. Then $z = \bigcup_{\alpha<\nu} z_\alpha \in C(f)$, since $\bigcup f“[z_\alpha]^{<\omega} \subset z_{\alpha+1}$. Since $\{z_\alpha : \alpha < \nu\} \subset T_\gamma$ is increasing, $z \in T_\gamma$. Since $\bigcup\{c_\xi : \xi \in z_\alpha\} \subset z_{\alpha+1}$, $\{c_\xi : \xi \in z\} \subset \mathcal{P}_\nu z$. To see that $\{c_\xi : \xi \in z\}$ is unbounded in $\mathcal{P}_\nu z$, fix $x \in \mathcal{P}_\nu z$. We have $\alpha < \nu$ with $x \subset z_\alpha$, hence $\xi \in z_{\alpha+1}$ with $x \subset c_\xi$, as desired. $\square$

Theorem 3 is void, however, if $\mathrm{cf}\,\mu < \kappa$ or if $\kappa = \theta^+$ and $\theta > \mathrm{cf}\,\theta = \omega$: In the former case $\mathcal{P}_\kappa\mu$ has no stationary subset of size μ. In the latter case $\mathcal{P}_\nu z$ has no unbounded subset of size θ for $z \in [\mu]^\theta$, since $\mathrm{cf}(\nu, \theta) > \theta$ if $\mathrm{cf}\,\theta < \nu < \theta$. See [9] for a nonreflection result in the latter case under additional assumptions.

4. Proof of Theorem 1

This section is devoted to Foreman–Magidor's example of a nonreflecting stationary set as we understand it. The proof invokes those [1, 2] that $\mathcal{P}_\kappa\kappa^+$ has a club subset of size $\leq (\kappa^+)^{\omega_1}$ and that stationary reflection implies Chang's conjecture.

Proof of Theorem 1. Fix a bijection $\pi_\gamma : \kappa \to \gamma$ for $\kappa \leq \gamma < \kappa^+$. Define $h : [\kappa^+]^2 \to \mathcal{P}_\kappa\kappa^+$ by $h(\alpha,\beta) = \lim \pi_\beta``{\pi_\beta}^{-1}(\alpha)$. Since $\lambda \geq 2^{\kappa^+}$, we have a list $\{g_\xi : \xi < \lambda\}$ of the functions $g : \kappa^+ \to \mathcal{P}_\kappa\kappa$. Then $D = \{x \in \mathcal{P}_\kappa\lambda : \bigcup h``[x \cap \kappa^+]^2 \subset x \wedge \forall\gamma \in x \cap (\kappa^+ - \kappa)(\pi_\gamma``(x \cap \kappa) = x \cap \gamma) \wedge \forall\xi \in x(\bigcup g_\xi``(x \cap \kappa^+) \subset x)\}$ is club in $\mathcal{P}_\kappa\lambda$.

Set $S = \{x \in \mathcal{P}_\kappa\lambda : \{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is nonstationary in $\kappa^+\}$.

Claim. *S is stationary in $\mathcal{P}_\kappa\lambda$.*

Proof. Suppose otherwise. By induction on $n < \omega$ build $f_n : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ and $\xi_n : [\lambda]^{<\omega} \to \lambda$ so that $C(f_0) \subset D - S$, $g_{\xi_{n}(a)}(\gamma) = \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa$ and $f_{n+1}(a) = f_n(a) \cup \{\xi_n(a)\}$. Define $f : [\lambda]^{<\omega} \to \mathcal{P}_\kappa\lambda$ by $f(a) = \bigcup_{n<\omega} f_n(a)$.

Subclaim. $\{\sup(z \cap \kappa^+) : x \subset z \in C(f) \wedge z \cap \kappa = x \cap \kappa\}$ *is unbounded in κ^+ for $x \in C(f)$.*

Proof. Fix $\alpha < \kappa^+$. Since $x \in C(f) \subset \mathcal{P}_\kappa\lambda - S$, $\{\sup(y \cap \kappa^+) : x \subset y \in D \wedge y \cap \kappa = x \cap \kappa\}$ is stationary in κ^+. Hence we have $x \subset y \in D$ with $y \cap \kappa = x \cap \kappa$ and $\alpha < \gamma \in y \cap \kappa^+$.

Set $z = \bigcup\{\mathrm{cl}_{f_n}(a \cup \{\gamma\}) : n < \omega \wedge a \in [x]^{<\omega}\}$. Then $\alpha < \gamma \leq \sup(z \cap \kappa^+)$. It is easy to see $x \subset z \in C(f)$. To see $z \cap \kappa \subset x \cap \kappa$, fix $\beta \in z \cap \kappa$. Then $\beta \in \mathrm{cl}_{f_n}(a \cup \{\gamma\}) \cap \kappa = g_{\xi_{n}(a)}(\gamma)$ for some $n < \omega$ and $a \in [x]^{<\omega}$. Since $x \in C(f)$ and $a \in [x]^{<\omega}$, $\xi_n(a) \in f(a) \subset x \subset y$. Hence $\beta \in g_{\xi_{n}(a)}(\gamma) \subset y \cap \kappa = x \cap \kappa$, as desired, since $\xi_n(a), \gamma \in y \in D$. $\square$

For $i = 0, 1$ build an increasing and continuous sequence $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ so that $x^i_\xi \cap \kappa = x^0_0 \cap \kappa \in \kappa$ has cofinality ω_1, $\sup(x^0_\xi \cap \kappa^+) \leq \sup(x^1_\xi \cap \kappa^+) < \sup(x^0_{\xi+1} \cap \kappa^+)$ and $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$ as follows: First take $x^0_0 \in C(f)$ with $x^0_0 \cap \kappa \in S^{\omega_1}_\kappa$. Subclaim allows us to take x^0_1 from $X = \{z \in C(f) : x^0_0 \subset z \wedge z \cap \kappa = x^0_0 \cap \kappa\}$ so that $\{\sup(z \cap \kappa^+) : z \in X\} \cap \sup(x^0_1 \cap \kappa^+)$ has size κ. Since $x^0_1 \cap \kappa^+$ has $< \kappa$ initial segments, we have $x^1_0 \in X$ as required above. The rest of the construction is routine.

Set $x^i = \bigcup_{\xi<\omega_1} x^i_\xi$. Then $x^i \in C(f)$, since $\kappa \geq \omega_2$ is regular and $\{x^i_\xi : \xi < \omega_1\} \subset C(f)$ is increasing. Also $\sup(x^i \cap \kappa^+) = \sup_{\xi<\omega_1} \sup(x^i_\xi \cap \kappa^+)$ has cofinality ω_1 and agrees with each other by $\sup(x^0_\xi \cap \kappa^+) \leq \sup(x^1_\xi \cap \kappa^+) < \sup(x^0_{\xi+1} \cap \kappa^+)$. Since $x^i, x^i_\xi \in C(f) \subset D$, we have $x^i \cap \gamma = \pi_\gamma``(x^i \cap \kappa) = \pi_\gamma``(x^0_0 \cap \kappa) = \pi_\gamma``(x^i_\xi \cap \kappa) = x^i_\xi \cap \gamma$ for $\gamma \in x^i_\xi \cap (\kappa^+ - \kappa)$. Since $x^1_0 \cap \kappa^+$ is not an initial segment of $x^0_1 \cap \kappa^+$, $x^i \cap \kappa^+$ disagrees with each other. Moreover $x^i \cap \kappa^+$ is σ-closed:

Fix $b \subset x^i \cap \kappa^+$ of order type ω. We have $b \subset \beta \in x^i \cap (\kappa^+ - \kappa)$ by $\mathrm{cf}\sup(x^i \cap \kappa^+) = \omega_1$. Since ${\pi_\beta}^{-1}``(x^i \cap \beta) = x^i \cap \kappa = x^0_0 \cap \kappa \in \kappa$

has cofinality ω_1, we have $\alpha \in x^i \cap \beta$ with ${\pi_\beta}^{-1}``b \subset {\pi_\beta}^{-1}(\alpha)$. Hence $b \subset \pi_\beta``{\pi_\beta}^{-1}(\alpha)$. Thus $\sup b \in h(\alpha, \beta) \subset x^i$, as desired, since $\alpha, \beta \in x^i \in D$.

Set $c = x^0 \cap x^1 \cap \kappa^+$, which is unbounded in $\sup(x^i \cap \kappa^+)$. Then $x^i \cap \kappa^+ = \bigcup_{\gamma \in c} x^i \cap \gamma = \bigcup_{\gamma \in c} \pi_\gamma``(x^i \cap \kappa) = \bigcup_{\gamma \in c} \pi_\gamma``(x^0_0 \cap \kappa)$ by $x^i \in D$, which contradicts that $x^i \cap \kappa^+$ disagrees with each other, as desired. $\square$

Claim. *S is nonreflecting.*

Proof. Suppose to the contrary $S \cap \mathcal{P}_\kappa A$ is stationary in $\mathcal{P}_\kappa A$ for some $\kappa \subset A \subset \lambda$ of size κ. Fix a bijection $\pi : \kappa \to A$. Then $T = \{\gamma < \kappa : \pi``\gamma \in S \wedge \pi``\gamma \cap \kappa = \gamma\}$ is stationary in κ, hence $\{y \cap \kappa^+ : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary in $\mathcal{P}_\kappa \kappa^+$. Thus $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa \in T\}$ is stationary in κ^+, hence so is $\{\sup(y \cap \kappa^+) : \pi``(y \cap \kappa) \subset y \in D \wedge y \cap \kappa = \gamma\}$ for some $\gamma \in T$. Thus $\{\sup(y \cap \kappa^+) : \pi``\gamma \subset y \in D \wedge y \cap \kappa = \pi``\gamma \cap \kappa\}$ is stationary in $\kappa^+$, which contradicts $\pi``\gamma \in S$, as desired. $\square$

Therefore stationary reflection for $\mathcal{P}_\kappa \lambda$ fails. $\square$

We remark that the same proof as above works if we replace "nonstationary" by "bounded" in the above definition of S.

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